Analysis of Knowledge:
What Should Mathematics Teachers Know?

By Dorothy Vásquez-Levy & Maria A. Timmerman

Introduction

Recent efforts to reform mathematics education have resulted in recommendations that teachers change their instructional practices and use alternative reform curricula to enable students to construct new knowledge of mathematical concepts and skills (National Council of Teachers of Mathematics [NCTM], 1989, 1991, 1995, 1998). While actively engaged in challenging tasks, teachers and students are asked to work together to answer meaningful exercises that involve logical and proportional reasoning and testing and connecting mathematical concepts. The NCTM *Standards* (1989, 1991, 1995, 1998) emphasize goals and value judgments about what students should know and be able to do. Being explicit about what one *values* as mathematical knowledge is crucial for informing teachers' assessment of students' knowledge of mathematics.

Yet, to what extent do mathematics teachers understand the nature of knowledge? Because their teaching decisions and actions influence the learning of many, they are obligated to exercise a high regard for

Dorothy Vásquez-Levy and Maria A. Timmerman are assistant professors in the Department of Curriculum, Instruction, and Special Education at the Curry School of Education, University of Virginia, Charlottesville, Virginia.

knowledge. There is consensus that a significant relationship exists between teachers' knowledge of mathematics, their conceptions of teaching, and their understandings of students' learning of mathematics (Fennema & Franke, 1992; Knapp & Peterson, 1995). However, little is known concerning the evolving nature of teachers' understanding of knowledge that will support changes in their conceptions and practices to be more in line with the goals of current mathematics education reform.

Current debate centers on preserving certain knowledge (e.g., teacher telling versus student inventing computation procedures) but also working to dismantle established regulations framing curriculum and standards of learning. Many mathematics teachers agree that national reform principles and state standards are important for students' development of mathematical thinking and understanding, but agreement ends there. Who should determine what mathematical knowledge and pedagogy should be implemented? What standards are most valued? How do teachers' conceptions of mathematics and mathematics teaching and learning contribute to decisions made about teaching practices? As Cooney (1984) claimed, "I believe that teachers make decisions about students and the curriculum in a rational way according to the conceptions they hold" (p. 89). Therefore, asking teachers to implement standards-based curricula and pedagogy does not happen overnight; it requires learning. Without in-depth, rich opportunities to assess their own understanding of mathematics; their students' mathematical knowledge; and participating in analytical reflection on their practice; many teachers will find it difficult to make sense of new mathematics curricula and teaching practices intended to change the way they teach.

When mathematics teacher educators examine types of knowledge in the context of teacher development, they are better positioned to plan, implement, and assess ways to improve learning opportunities for teachers. We suggest that if teacher educators attend to their own understandings of the nature of knowledge, they can more readily interpret and assess the growing development of teachers' understandings of knowledge. Furthermore, they can facilitate teachers' reflective thinking about the nature of knowledge and how knowledge may be accepted when teachers assess students' understanding of mathematics.

In this article, we discuss types of knowledge and the specific conditions which warrant them. To achieve this end, we discuss concepts from critical epistemology as a preliminary analysis. We suggest a series of questions for a critical practice of teaching that takes knowledge as information and the decision making of teaching seriously.

Of Epistemological Perspective: The Nature of Knowledge

Asking what we know is the work of epistemologists, but it is also the task of all teachers who make decisions about the meaning of students' thinking on knowing and applying mathematics in their classrooms. The work begins with common sense and scientific assumptions about what is real and what is known. These convictions constitute our data, possibly even conflicting data if common sense and science conflict. For example, some teachers gather data by observing and listening to students construct their own ideas and meaning for specific mathematical topics which often runs counter to many teachers' own learning experiences of memorizing facts and procedures transmitted by their teachers. The object of making meaning of students' mathematical thinking, of which a systematic and critical analysis of prior information and practice is a component, is to account for the data. Sometimes teachers explain the data, and sometimes they explain the data away. For the most part, making meaning requires teachers to construct an explanation for how we know the things we know, but in a few instances, it also requires an explanation for why we think we know when we do not.

To explain what we do know and why we do not, however, we should first ask what the knowledge is. Indeed, given the complexity of different aspects of teachers' knowledge and their relationship to the ill-defined nature of school classroom environments within which teachers work, it is inexcusable for teachers to accept any knowledge without first analyzing it and acquiring meaning for the knowledge presented to them.

What is Knowledge? A Western Conception

People have knowledge. But what kind of knowledge do they have, and what is knowledge anyway? There are many ways of knowing, but only one knowledge for saying that something is true. Consider the following sentences:

1. I know calculators.
2. I know ways to draw triangles.
3. I know the traditional computational algorithm for adding two three-digit numbers.
4. I know algebra.
5. I know heuristics for solving problems.
6. I know the standards of learning in mathematics education.
7. I know that what you say is true.
8. I know the mathematical concept of 10 indicates that a 10 is the same as 10 ones.
9. I know the Moebius strip is a one-sided surface that has many unexpected properties.
10. I know the sentence "I can construct physical or conceptual models of phenomena" is true.

These are but a few samples of different uses of the word *know* describing different sorts of knowledge. If we are interested in discovering what people have when they have knowledge, we must first sort out the varied senses of the word *know*. Then we may ask our question again: What *is* knowledge, once it has been disambiguated?

In one respect, "to know" means to have some form of special competence. To know calculators or to know multiplication tables up to 12 times is to be competent to use calculators or to recall the products of any two numbers not exceeding 12. If a person is said to know how to do something, it is this competent sense of "know" that is usually involved. If I say I know ways to draw triangles, I mean that I have attained the special kind of competence needed to recall the necessary and sufficient conditions of its shape. If I say I know the traditional computational algorithm for adding two three-digit numbers, I mean that I have the special competence required to recall or to recite the characteristics of its occurrence.

Another sense of "know" is that in which the word means to be acquainted with something or someone. When I say that I know algebra, I mean that I am familiar with the subject or acquainted with the content with this name. The sentence "I know heuristics for solving problems" is more difficult to disambiguate. It might mean simply that I am acquainted with heuristics and, therefore, have the acquaintance sense of "know," or it might mean that I have the special form of competence needed to strategize and solve problems, mathematically and/or socially. It also might mean that I know it in both the competence and acquaintance sense of "know." This example illustrates the important fact that the senses of "know" we are distinguishing are not exclusive; therefore, the term *know* may be used in more than one of these senses in a single utterance.

The third sense of "know" is that in which "to know" means to recognize something as information. If I know that the mathematical concept of 10 indicates that a 10 is the same as 10 ones, then I recognize something as information, namely, the concept of 10 indicates that a 10 is the same as 10 ones. The last three sentences on the list all involve this information sense of the word *know*. It is often affirmed that to know something in the other senses of "know" entails knowledge in the information sense of "know." I must have some information about calculators if I know how to use one, about triangles if I know how to draw them, about computational algorithms if I know the procedures, about heuristics if I know them, and so forth. Therefore, the information sense of the word *know* is often implied in the other senses of the word.

In this discussion, we are concerned with knowledge in the information sense. It is precisely this sense that is fundamental to human thought and required for theoretical speculation and practical judgment. To make decisions about the meaning of students' mathematical thinking, teachers require knowledge in the information sense. This kind of knowledge goes beyond the mere possession of information. If you tell me something and I believe you, even though I have no idea whether you are a source of truth and have correct information about the subject or are a disseminator of falsehood and deception, I may, if I am fortunate, acquire information. This is not, however, knowledge in the sense that concerns us; it is merely the possession of information. Similarly, if I read the local newspaper and believe the information I receive, though I have no idea whether the paper is reporting accurately, I may acquire information, but this is not knowledge. For example, in what is referred to as the California math wars, writers of the popular press broadly disseminate criticisms of the mathematics education reform movement by non-educators (mathematicians, scientists, parents, and other citizens) (Schoen, Fey, Hirsch, & Coxford, 1999). Such ignorance of the reliability of the source prevents us from recognizing that the information is correct, from knowing that it is correct, even though we may believe it to be. It is the information that we recognize to be genuine that yields the characteristically human sort of knowledge that distinguishes us as adult thinking beings.

The generation of new ideas emerges from our ability to see and interpret things in new ways, stemming from the creativity of the human mind. Our most valuable scientific achievements, the discovery of the double helix, for instance, and our most worthy practical attainments, such as the unfolding of a system of justice, depend on a more significant kind of knowledge. This kind of knowledge rests on our capacity to distinguish truth from error.

Analyzing Selected Knowledge

To indicate the information sense of the word *know* as being the one in question is quite different from analyzing the kind of knowledge we have selected. What is an analysis of knowledge? An analysis is always relative to some objective. It does not make sense simply to demand the analysis of knowledge, truth, goodness, or beauty without some indication of what purpose such an analysis is intended to achieve. To demand the analysis of knowledge without specifying further what you hope to accomplish with it is like demanding standards of learning without defining what kind of persons you hope students will become. Before asking for such an analysis, we should make clear what goals we hope to achieve with it.

Useful Questions

When teachers are presented with knowledge in the information sense it is always helpful to ask different types of questions. So far, we have emphasized four important questions: (1) What sense of the word *know* am I using?, (2) In what ways is the information sense of the word *know* implicated in the other senses of the word?, (3) What is the reliability of the source of this knowledge?, and (4) How do I distinguish knowing that the information is correct from believing it to be the case?

Another useful approach to verifying knowledge is to seek counterexamples to what is presented in a teacher's decision. For example, teachers should provide learning experiences that engage students in developing *proportional reasoning* because it is a "gatekeeper" to understanding high-school-level mathematics and science courses (e.g., algebra, geometry, biology, chemistry, and physics). Successful completion of these courses can lead to future careers in mathematics and science. According to NCTM (1998),

> Proportional reasoning permeates the entire middle grades' curriculum, is a key integrative thread that connects most of the mathematics topics studied in the middle grades, and can be developed in several areas of mathematics. (p. 213)

Students need many learning opportunities to develop a rich, deep understanding of proportional relationships that exist in school and everyday situations involving fractions, decimals, percentages, ratios, measurements, and similarity if they are to make well-grounded decisions about using information. Otherwise, they will join the "more than half the adult population [who] cannot reason proportionally" (Lamon, 1999, p. 5).

As an example of a teacher focusing on developing her sixth-grade students' understanding of proportional reasoning, Mrs. Kelly gave her students the following problem about the growth of two snakes (Lamon, 1999).

> Jo has two snakes, String Bean and Slim. Right now, String Bean is 4 feet long and Slim is 5 feet long. Jo knows that two years from now, both snakes will be fully grown. At her full length, String Bean will be 7 feet long, while Slim's length when he is fully grown will be 8 feet. Over the next two years, will both snakes grow the same amount? (p. 12)

One goal for solving this problem is to have students use mathematical reasoning rather than applying rules or using a proportion equation (e.g., $a/b=c/d$). We invite readers to solve the problem before reading on and to compare your responses to those of different groups in the class.

Students' solutions to the problem tend to fall into two categories: those that use absolute or additive thinking and those that use relative or multiplicative thinking. Using absolute or *additive thinking*, the snakes will grow the same amount because the change in the length of each snake is 3 feet. Actual growth is not compared to or related to anything else. In contrast, using relative or *multiplicative thinking*, the expected growth of each snake can be compared to its present length, that is, String Bean will grow 3 feet which is 3/4 of her present length of 4 feet and Slim will grow 3 feet which is 3/5 of his present length of 5 feet. Because the relative change in growth represented by the fraction 3/4 is greater than the fraction 3/5, String Bean will grow more than Slim. Mrs. Kelly encouraged student discussion about the problem so that all the groups would analyze the problem from these two different perspectives: absolute and relative change.

As the students worked in groups, they drew pictures to represent the beginning lengths of the two snakes, 4 feet for String Bean and 5 feet for Slim, and final lengths of 7 feet and 8 feet, respectively, for the two snakes. Most of the groups mentally calculated the difference between the beginning and final lengths of the snakes which resulted in a growth of 3 feet for each snake. Some groups wrote down "3 feet" next to each of their representations of the snakes and shared their answer, that the snakes would "grow the same amount" over the next two years, with the class. Even though this answer was correct, Mrs. Kelly realized that many of the students only relied on very familiar additive thinking that they had practiced for several years in elementary school.

When the teacher asked if any group had solved the problem in a different way, Sonia's group claimed that they had come up with two answers depending on how you solved the problem. They agreed with most of the other groups and said one answer would be that both of the snakes grew the same amount. A second answer was that "Slim would grow more than String Bean" because 3/5 was more than 3/4. Looking at their drawings of the initial lengths of the snakes and the final lengths in two years, the teacher noticed some numbers and calculations Sonia's group had written down. For String Bean, "3 feet," "4 feet," and "3/4 feet" appeared as some of their work for solving the problem. Next to Slim, the group wrote "3 feet," "5 feet," and "3/5 feet." Below the drawings, Sonia's group wrote two inequalities: "5>4" and "3/5>3/4." Without verification, Mrs. Kelly could decide that this group of students had used multiplicative thinking because the final inequality (i.e., 3/5>3/4) did use the correct fractions to compare the relative change in growth; yet, either they did not understand how to reason about the size of fractions or they used an incorrect inequality symbol.

Looking to verify her understanding of the students' solution, Mrs. Kelly challenged Sonia to explain how her group solved the problem. Sonia explained that they first calculated "in our heads" how much each snake grew individually over the two-year period. Like many of the other groups, they determined an absolute change of 3 feet for each snake, so the snakes would grow the same amount. Then, Sonia explained that one of the group members said that Slim grew more because he was bigger in the beginning. In attempting to verify if the group had indeed used multiplicative thinking, Mrs. Kelly questioned Sonia about what the final inequality, 3/5>3/4, meant in their written work. After some hesitation, Sonia explained that in comparing the initial lengths of the snakes, Slim's 5 feet was greater than String Bean's 4 feet, so they wrote "5>4." Next, they decided to use the 3 feet from their first answer and wrote their second answer, "3/5>3/4" because they had recently used fractions to solve other problems. Also, they decided to keep the greater than symbol because both fractions had numerators of three, so they only had to look at the denominators of the fractions. Because the "five" in 3/5 was greater than the "four" in 3/4, Sonia's group decided that "3/5>3/4" which would result in Slim growing more than String Bean. Moreover, their comparison of the fractions affirmed one of the group member's belief that Slim grew more than String Bean because he started out as the bigger snake. By listening to Sonia's reasoning, the teacher verified that this group of students had not used multiplicative thinking even though the "correct" fractions appeared in their work. Although they seemed to understand how to use inequality symbols, they made connections to their familiar whole number knowledge rather than rational number knowledge when comparing the size of the fractions.

Any experiment of fact or thought that would falsify the resulting equivalence is a counterexample. Begin by considering any logically possible case as a potential counterexample to the knowledge before you. It may be that some examples, though logically possible, are so remote in terms of real possibility that they do not constitute realistic objections to any analysis of knowledge.

It is important to consider from the onset what a teacher's informed decision is attempting to explain, hence the analysis. Aside from the responsibility, making and analyzing meaning of students' mathematical thinking also carries obligations to act. One obligation is to ensure that the representation the meaning from thinking takes as it informs people is correct information rather than error and misinformation. Teachers should be certain that they have all the information. Such an obligation requires one to make certain the information is accurate and not simply believe what he or she observes, but "to know that" the meaning-making process is correct. Similarly, if in the work of analyzing the meaning of students' mathematical problem solving, you possess some information in memory in relation to the thinking but no longer know whether it is correct information, whether it is something you accurately remember, or just something imagined, you are ignorant of the matter. If, on the other hand, you have verified whether the meaning is correct, then you can "know that" the information possessed is correct.

One way to test your "knowing that" the information you possess is correct is to determine whether you can answer the question of how you know that the information is correct or how you would justify claiming to know. These are the critical questions, and the answers are the basis for critical discussion and rational argument in teaching, scientific inquiry, and everyday life. The responses to these questions indicate whether the conditions for knowledge have been met. If a person claims to know something, how well he or she answers the question "How do you know?" will determine whether his or her claim is accepted. Consequently, the analysis of knowledge should demonstrate how a person knows that his or her information is correct and how his or her knowledge claims are justified. Such a test requires knowing agents—individuals who understand the difference between truth and correct information on the one hand and deception and misinformation on the other.

In summary, teachers can be more understanding of knowledge by asking certain types of questions. For example, to understand the knowledge in use, begin by asking,

1. What is the sense of the word *know* I am using?
2. In what ways is the information sense of the word *know* implicated in the other senses of the word?
3. What is the reliability of the source of this knowledge?
4. How do I distinguish knowing that the information is correct from believing it to be the case?
5. Are there counterexamples to what is being presented in the meaning of

students' mathematical thinking? Any experiment of fact or thought that would falsify the resulting equivalence is a counterexample.

6. How do I know that the knowledge in the information sense is correct?

Further, it is important to raise these questions in the context of mathematics teachers' professional development because they engage teachers in clear articulation of what they value as types of mathematical knowledge in use to assess the depth of students' understanding of mathematics. Such questions examine how teachers know whether the knowledge being evaluated is correct and reliable.

One goal of professional development is to encourage teachers' discussion and analytical reflection on the nature of knowing when assessing the understandings of students. Discussion can raise issues related to a developmental construction of mathematics in which we move beyond a binary notion of those who either "have it" or "do not have it" in terms of knowing mathematics. These questions can also enable mathematics teacher educators to address their own understandings of the nature of knowing when interpreting and assessing the depth of teachers' understanding of mathematics and the teaching and learning of mathematics.

Next, we turn to discussion of what makes knowledge warranted.

Conditions of Knowledge

Our intent in this section is to provide a thumbnail sketch of the three conditions for knowledge. The first is truth. Something is true if and only if the truth is known. For instance, it is true that the California State Board of Education adopted the new California Mathematics Standards (1997) if and only if the California State Board of Education did adopt the new California Mathematics Standards. The second condition of knowledge is what Lehrer (1990) called acceptance. If we deceitfully claim to know that mathematical instruction is no longer teacher-directed explanations of procedures but rather student-invented algorithms facilitated by teachers when we do not accept it, then we do not know it even if this is what is stated. If we do not accept that knowledge, then we do not know that knowledge. "It is the acceptance of something in the quest for truth that is the required condition of knowledge" (Lehrer, 1990, p. 11). Furthermore, a person need not have a strong feeling that something is true to know that it is. What is required is acceptance of the appropriate kind, acceptance in the interest of obtaining a truth and avoiding an error in what one accepts (Lehrer, 1990). The terms that should always be introduced when precision is needed in analyzing a knowledge claim are *accept* and *acceptance*.

The final condition for knowledge is justification, and its purpose is the attainment of truth. Truth is what propels the engine of justification. How are we to decide whether what is suggested to us by our senses is true and accurate rather than false and illusory? We have to consult information about the matter. What is this information? It is what we have accepted in our search for the truth. Acceptance is what stimulates the engine of justification. Our acceptance system is the tool we use whenever we have to decide what to accept on the basis of the information we currently hold. It is our repository of information about the world, and it is the basis for making judgments. We have to decide how reasonable it is to accept new information in comparison to other competing factors. Now, in response to new data and further deliberation, one's acceptance system changes. If we find the information to be more trustworthy in terms of source and situation than conflicting or undermining objections, then it is more reasonable for us to accept the information on the basis of that system because of the way it relates with that system. That is the way coherence renders in justification. Our ability to verify information is crucial to this notion of coherence. Because we are human, we are capable of making mistakes in our judgments. Therefore, it is important that we not only call upon our acceptance system but simultaneously correspond the information we accept with evidentiary data. As Lehrer (1990) suggested, "Knowledge, or undefeated justification, results from the right combination of coherence, acceptance, and truth" (p. 151). Put another way, we accept what we do with the objective of attaining truth and avoiding error.

Conclusion

Our development of the analysis of knowledge and the specific conditions for warranting it offers a particular conceptual lens that frames our thinking about teaching and what mathematics teachers should know. A challenge before us as mathematics teachers is determining whether our justification is complete and undefeated. What is required of us is to ensure that the judgments we render are well-grounded. This alone will determine whether we succeed in our attempt to make sound decisions. If we succeed in our efforts without acting on error, our justification for making meaning of students' mathematical thinking is complete and undefeated, and we gain knowledge. If our decisions rest on error, we have lost the prize of knowledge. We argue that mathematics teacher educators' understandings about the nature of knowledge and its relationship to teacher development have the potential to contribute to teachers shifting away from acquired or superficial aspects of knowledge toward acceptance or more significant aspects of knowledge.

A Final Note

For those interested in reading literature on the theory of knowledge, there are a number of good introductory texts. Paul K. Moser and Arnold Vander Nat (1987) edited a general collection of essays called *Human Knowledge*. This text presents ideas from both the classical and contemporary literature. George S. Pappas and Marshall Swain (1978) edited a collection of contemporary articles on the nature of knowledge *Essays on Knowledge and Justification*. Two readable traditional works are *The Problems of Philosophy* by Bertrand Russell (1959) and *The Problem of*

Knowledge by Alfred J. Ayer (1957). Among the helpful texts by individual authors are: *Contemporary Theories of Knowledge* by John L. Pollock (1986); *Belief, Justification, and Knowledge* by Robert Audi; *Theory of Knowledge* (3rd ed.) by Roderick Chisholm (1989); and *Theory of Knowledge* by Keith Lehrer (1990).

For those interested in reading literature on the theory of mathematics, see *The Philosophy of Mathematics Education* by Paul Ernest (1991), and *What is Mathematics, Really?* by Reuben Hersh (1997).

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